NORTH SYDNEY BOYS HIGH SCHOOL

EXTENSION 2 MATHEMATICS

2023 HSC Course Assessment Task 3

June 23, 2023

General instructions

- Working time – 3 hours. (plus 10 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided

SECTION II

- Commence each new question on a NEW BOOKLET.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Teacher (please ✔)

○ Mr Berry

○ Ms Lee

○ Mr Umakanthan

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	**Total**
MARKS	/10	/15	/15	/15	/16	/15	/14	/100

Section I **Marks**

Question 1 **1**

The converse of $P \Rightarrow Q$ is:

A $Q \Rightarrow P$

B $Q \Leftrightarrow Q$

C $(\text{not } P) \Leftrightarrow (\text{not } Q)$

D $(\text{not } Q) \Rightarrow (\text{not } P)$

Question 2 **1**

Suppose $z = p + iq$ is a solution of the polynomial equation

$$c_4 z^4 + ic_3 z^3 + c_2 z^2 + ic_1 z + c_0 = 0$$

where p, q, c_4, c_3, c_2, c_1 and c_0 are real.

Which of the following must also be a solution?

A $q + ip$

B $p - iq$

C $-p - iq$

D $-p + iq$

Question 3 **1**

Vectors $\mathbf{u}$ and $\mathbf{v}$ have components $\begin{pmatrix} 0.6 \\ 0 \\ t \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 0 \\ -5 \end{pmatrix}$ respectively.

The following two statements are made about $\mathbf{u}$ and $\mathbf{v}$:

(1) when $t = -1$, $\mathbf{u}$ and $\mathbf{v}$ are parallel

(2) when $t = -0.8$, $\mathbf{u}$ is a unit vector

Which of the following is true?

A Neither statement is correct.
B Only statement (1) is correct.
C Only statement (2) is correct.
D Both statements are correct.

Marks

Question 4 **1**

Which of the following has a converse that is not always true?

A If a triangle is a right triangle, then the square of the longest side of the triangle is equal to the sum of the squares of the other two sides

B If the shape is a triangle, then the sum of all three interior angles is $180°$

C If the shape is a square, then it is also a rectangle

D If two parallel lines are intersected by a transversal, then the alternate interior angles, alternate exterior angles, and the corresponding angles are congruent

Question 5 **1**

A particle is moving in Simple Harmonic Motion in a straight line with amplitude 8m.
The speed of the particle is 12m/s when it is 4 metres from the centre of its motion.
What is the period of the motion?

A $\frac{2\pi}{3}$

B $\frac{2\sqrt{3}\pi}{3}$

C $\sqrt{3}$

D 3

Question 6 **1**

Which integral is necessarily equal to

$$\int_{-a}^{a} f(x)dx\,?$$

A $\displaystyle\int_{-a}^{a} [f(x) - f(-x)]dx$

B $\displaystyle\int_{0}^{a} [f(x) - f(a - x)]dx$

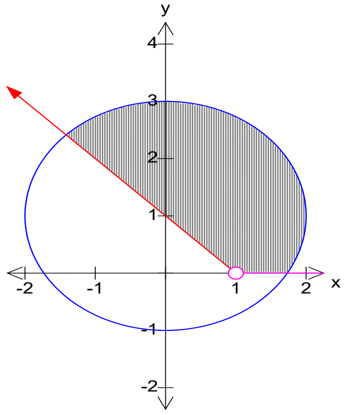
C $\displaystyle\int_{0}^{a} [f(x - a) + f(-x)]dx$

D $\int_0^a [f(x-a)+f(a-x)]dx$

Marks

Question 7 **1**

Consider the Argand diagram below.



Which inequality could define the shaded area?

A $|z-i|\leq 2$ and $0\leq \arg(z-1)\leq \frac{3\pi}{4}$

B $|z+i|\leq 2$ and $0\leq \arg(z-1)\leq \frac{3\pi}{4}$

C $|z-i|\leq 2$ and $0\leq \arg(z-1)\leq \frac{\pi}{4}$

D $|z+i|\leq 2$ and $0\leq \arg(z-1)\leq \frac{\pi}{4}$

Question 8 **1**

Which of the following is an expression for $\int \frac{2}{x^2+4x+13}dx$?

A $\frac{1}{3}\tan^{-1}\frac{(x+2)}{3}+c$

B $\frac{2}{3}\tan^{-1}\frac{(x+2)}{3}+c$

C $\frac{1}{9}\tan^{-1}\frac{(x+2)}{9}+c$

D $\frac{2}{9}\tan^{-1}\frac{(x+2)}{9}+c$

Question 9 **1**

What are the zeros of the equation $x^4+x^2+6x+4=0$ over the complex field given that it has a rational zero of multiplicity 2?

A -1, $1+\sqrt{5}i$ and $1-\sqrt{5}i$

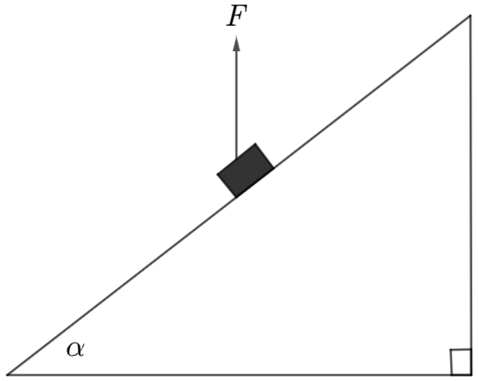
B -1, $1+\sqrt{3}i$ and $1-\sqrt{3}i$

C $+1$, $1+\sqrt{5}i$ and $1-\sqrt{5}i$

D $+1$, $1+\sqrt{3}i$ and $1-\sqrt{3}i$

Marks

Question 10 **1**



An object of mass $'m'$ is kept in equilibrium on a fixed incline at an angle $\alpha = 37°$ with the help of a force F acting vertically as shown in the diagram.
Friction (f) is the force that resists motion when the surface of one object comes in contact with the surface of another.
If there exists friction $f = F$ on the surface of the plane, which of the following is the closest value of F that is required to keep the object in equilibrium when the thread is held vertically?

A) $\dfrac{mg}{8}$

B) $\dfrac{2mg}{7}$

C) $\dfrac{3mg}{5}$

D) $\dfrac{3mg}{8}$

Section II

Question 11 (15 Marks) **Marks**

a) If $z_1 = 1 + \sqrt{3}i$ and $z_2 = \sqrt{3} - i$, find, expressing in the form $e^{i\theta}$

i $\left(\frac{z_1}{z_2}\right)^{79}$ **2**

ii $\frac{(z_1)^{10}}{(z_2)^8}$ **2**

b) Evaluate **4**

$$\int_{-2}^{0} \frac{8}{(x-2)(x^2+4)}\,dx$$

c) Find the two complex numbers which satisfy: **3**

$$3z\bar{z} + 2(z - \bar{z}) = 39 + 12i$$

d) The complex number $z = x + iy$ is such that $\frac{z-8i}{z-6}$ is purely imaginary. Find the equation of the point P representing z and clearly graph on an Argand diagram **4**

Question 12 (15 Marks) **Marks**

a) i Show that $a^2 + b^2 > 2ab$, where a and b are distinct positive real numbers **1**

ii Hence or otherwise, show that $a^2 + b^2 + c^2 > ab + bc + ca$, where a, b and c are distinct positive real numbers **2**

iii Hence, or otherwise, prove that **2**

$$\frac{a^2b^2 + b^2c^2 + c^2a^2}{a+b+c} > abc$$

b) Three forces act on an object which moves with constant velocity $\underset{\sim}{v} = \left(3\underset{\sim}{i} - 2\underset{\sim}{j}\right)\ m/s$. Two of the forces are $(3\underset{\sim}{i} + 5\underset{\sim}{j} - 6\underset{\sim}{k})$ Newtons and $(4\underset{\sim}{i} - 7\underset{\sim}{j} + 2\underset{\sim}{k})$ Newtons. Find the third force. **2**

c) A rock of mass 5 kg is propelled vertically upwards into the air from the ground with an initial velocity of 12 m/s. The rock is subject to a downward gravitational force of 50 Newtons and air resistance of $\frac{v^2}{2}$ Newtons in the opposite direction to the velocity $v\ m/s$

i Make a neat sketch showing the forces acting and the rock. **1**

Hence show that the equation of motion of the rock is $\ddot{x} = -\frac{v^2}{10} - 10$

ii Find the time taken for the rock to reach its maximum height **3**

iii Show that $v^2 = 244e^{-\frac{x}{5}} - 100$ **3**

iv Find the maximum height reached by the rock **1**

Question 13 (15 Marks) — Marks

a) Prove $(n+2)^2 - n^2$ is a multiple of 8 if and only if n is odd **3**

b) i Show that **1**

$$\sqrt{\frac{8-x}{x}} = \frac{4-x}{\sqrt{8x-x^2}} + \frac{4}{\sqrt{8x-x^2}}$$

ii Hence evaluate **3**

$$\int_0^2 \sqrt{\frac{8-x}{x}}\,dx$$

c) Let α be the complex root of $z^7 = 1$ with smallest positive argument.

i Show that **2**

$$1 + \alpha + \alpha^2 + \alpha^4 + \alpha^5 + \alpha^6 = 0$$

ii If $x^3 + ax^2 + bx + c = 0$ is a cubic equation with roots $\alpha + \alpha^6, \alpha^2 + \alpha^5$ and $\alpha^3 + \alpha^4$, find the values of a, b and c **3**

d) Disprove by means of a counterexample the statement below, where p and q are real numbers: **1**

If p and q are irrational, then $p + q$ is irrational

e) Prove by contradiction the statement below, where p and q are real numbers. **2**

If pq is irrational, then at least one of p and q is irrational

<u>Question 14 (16 Marks)</u> **Marks**

a) Find $\int \frac{\log x}{x} dx$ **1**

b) The tangent from the point $P(1, \sqrt{6}, 3)$ to the sphere $x^2 + y^2 + z^2 = 1$ intersects the sphere at the point C. Find $|\overrightarrow{PC}|$ **2**

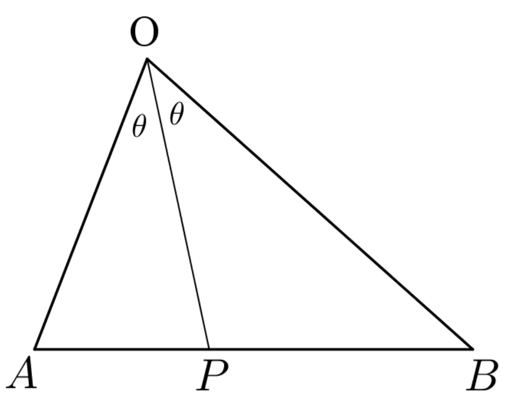
c)i If $I_n = \int x^n \cos x \, dx$, show that **3**

$$I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2}$$

ii Hence, or otherwise, evaluate **3**

$$\int_0^{\frac{\pi}{2}} x^4 \cos x \, dx$$

d)



In ΔOAB the point $P(a, b, c)$lies on AB such that OP bisects the angle at O and P divides AB in the ratio $m: n$

OP is extended to C such that $AO \parallel CB$.

If $O = (0,0,0), A\ (1,3,2), B(4,2,-6)$

i) Find the value of $\frac{m}{n}$ **2**

ii) Hence or otherwise, find the coordinates of P **1**

e) i Prove that **2**

$$\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta} = \sin\theta + i\cos\theta$$

ii Hence find the smallest value of θ such that **2**

$$(1 + \sin\theta + i\cos\theta)^5 + i(1 + \sin\theta - i\cos\theta)^5 = 0$$

Question 15 (15 Marks) **Marks**

a) An elevator filled with passengers has a mass of $1.70 \times 10^3\ kg$. The elevator accelerates upward from rest at a rate of $1.20\ m/s^2$ for $1.5\ s$. Take the value of acceleration due to gravity to be $9.8 m/s^2$

i Calculate the tension in the cable supporting the elevator. **2**

ii How high has the elevator moved above its original starting point, and what is its velocity at $t = 1.5s$? **3**

b) Raymond and Jordan have model airplanes, which take off from level ground. Jordan's airplane takes off after Raymond's. Distances are measured in metres.

The position of Raymond's airplane t seconds after it takes off is given by

$$\underset{\sim}{r_1} = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$$

i Find the speed of Raymond's airplane **1**

ii Find the height of Raymond's airplane after two seconds **1**

iii The position of Jordan's airplane s seconds are it takes off is given by **1**

$$\underset{\sim}{r_2} = \begin{pmatrix} -39 \\ 44 \\ 0 \end{pmatrix} + s \begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix}$$

Show that the paths of the airplanes are perpendicular

iv The two airplanes collide at the point $(-23, 20, 28)$. **3**

How long after Raymond's airplane takes does Jordan's airplane take off?

c) Convergent infinite series C and S are defined by

$$C = 1 + \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta + \cdots$$

$$S = \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \cdots$$

i Show that **2**

$$C + iS = \frac{2}{2 - e^{i\theta}}$$

ii Hence show that **2**

$$C = \frac{4 - 2cos\theta}{5 - 4cos\theta}$$

Question 16 (14 Marks) **Marks**

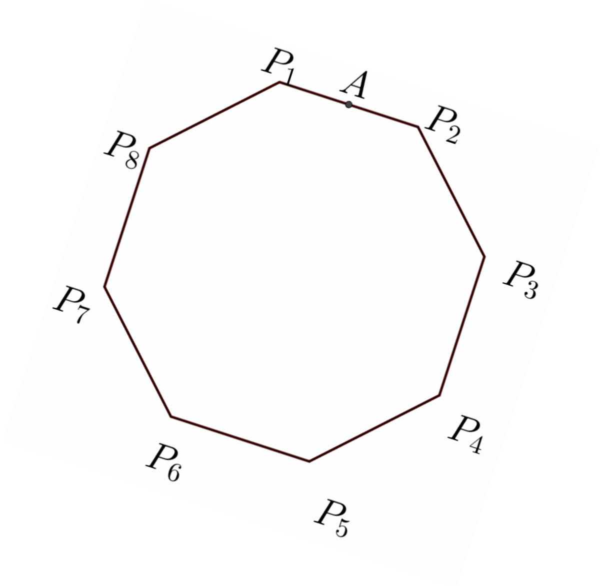
a) i Prove the result $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **1**

ii Hence, or otherwise, evaluate **3**

$$\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right) dx$$

b) In the diagram, $P_1P_2P_3P_4P_5P_6P_7P_8$ is regular octagon with side length 4. A is the midpoint of P_1P_2. **4**

Find the value of $|\overrightarrow{AP_1} + \overrightarrow{AP_2} + \overrightarrow{AP_3} + \overrightarrow{AP_4} + \overrightarrow{AP_5} + \overrightarrow{AP_6} + \overrightarrow{AP_7} + \overrightarrow{AP_8}|$



c) A particle is moving in simple harmonic motion of period T about a centre 0. Its displacement at any time t is given by $x = A\sin nt$, where A is the amplitude.

i Draw a neat sketch of one period of this displacement-time equation, showing all intercepts **1**

ii Show that $\dot{x} = \frac{2\pi A}{T}\cos\frac{2\pi t}{T}$ **1**

iii The point P lies D units on the positive side of O. Let V be the velocity of the particle when it first passes through P. Show that the time between the first two occasions when the particle passes through P is $\frac{T}{\pi}\tan^{-1}\frac{VT}{2\pi D}$ **4**

End of Examination

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Q1 A
Q2 D
Q3 D
Q4 C
Q5 B
Q6 D
Q7 A
Q8 B
Q9 B
Q10 D

Q11 (a) $z_1 = 1+\sqrt{3}i = 2e^{i\pi/3}$

$z_2 = \sqrt{3}-i = 2e^{-i\pi/6}$

(i) $\left(\frac{z_1}{z_2}\right)^{79} = \left(\frac{e^{i\pi/3}}{e^{-i\pi/6}}\right)^{79} = \left[e^{i(\pi/3+\pi/6)}\right]^{79} = \left(e^{i\pi/2}\right)^{79} = (i)^{79} = \cancel{-i}$

$= e^{-i\pi/2}$

(ii) $\frac{(z_1)^{10}}{(z_2)^8} = \left(\frac{z_1}{z_2}\right)^8 \cdot z_1^2 = (i)^8 (2e^{i\pi/3})^2 = \cancel{4\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}$

$4e^{i2\pi/3} = 2+2\sqrt{3}i$

(b) $$\int_{-2}^{0} \frac{8}{(x-2)(x^2+4)}\,dx$$

$$= \int_{-2}^{0} \frac{1}{x-2} - \frac{x}{x^2+4} - \frac{2}{x^2+4}\,dx$$

$$= \left[\ln|x-2| - \frac{1}{2}\ln(x^2+4) - \tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{0}$$

$$= \left(\ln 2 - \frac{1}{2}\ln 4 - 0\right) - \left(\ln 4 - \frac{1}{2}\ln 8 + \pi/4\right)$$

$$= -\left(\frac{1}{2}\ln 2 + \pi/4\right)$$

(c) Let $z = x+iy$

$3z\bar{z} + 2(z-\bar{z}) = 39+12i$

$\Rightarrow 3(x^2+y^2) + 4iy = 39+12i$

Matching the real parts $x^2+y^2 = 13$
.. .. im. parts $4y = 12$

$\Rightarrow \left.\begin{matrix} x=2 \\ y=3 \end{matrix}\right\}$ $\left.\begin{matrix} x=-2 \\ y=3 \end{matrix}\right\}$

$\therefore z = 2+3i$ or $z = -2+3i$

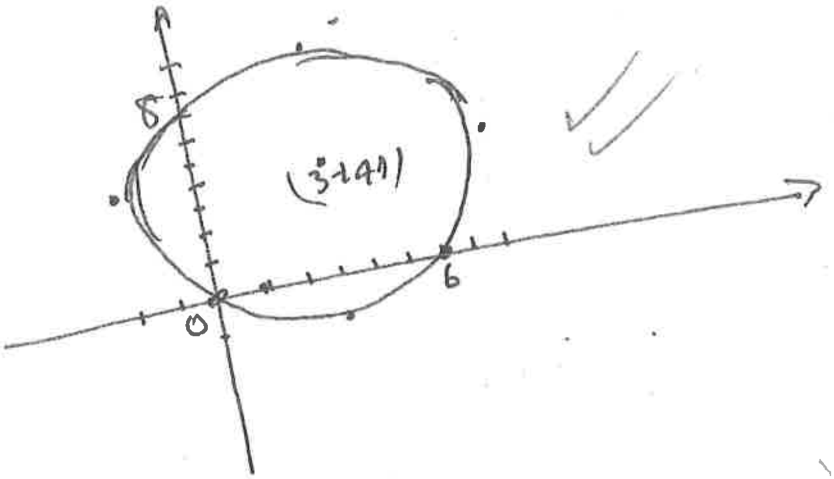
(d) $\frac{z-8i}{z-6}$ is purely imaginary $\Rightarrow Re\left(\frac{z-8i}{z-6}\right) = 0$

$\Rightarrow Re\left(\frac{x+(y-8)i}{(x-6)+iy}\right) = 0 \Rightarrow x(x-6)+y(y-8) = 0$

$\Rightarrow x^2+y^2-6x-8y = 0$

$\Rightarrow (x-3)^2+(y-4)^2 = 5^2$

11(d) $\therefore$ The locus of P is a circle with centre at $(3+4i)$ and radius 5 units



12a.

i) $(a-b)^2 > 0$

$a^2 - 2ab + b^2 > 0$

$a^2 + b^2 > 2ab \quad (1)$

ii) Similarly, $b^2 + c^2 > 2bc \quad (2)$

$a^2 + c^2 > 2ac \quad (3)$

$(1) + (2) + (3)$:

$2(a^2+b^2+c^2) > 2(ab + bc + ac)$

$a^2+b^2+c^2 > ab + bc + ac$

iii) W.L.O.G let $a^2 \to a^2b^2$

$b^2 \to b^2c^2$

$c^2 \to c^2a^2$

$a^2b^2 + b^2c^2 + a^2c^2 > ab^2c + bc^2a + a^2bc$

$= abc\,(b + c + a)$

$\therefore \dfrac{a^2b^2 + b^2c^2 + a^2c^2}{(a+b+c)} > abc \quad (\text{since } a+b+c \geq 0)$

12b Constant velocity: $\ddot{x} = 0$

$\Sigma F = 0$

Let the third force be $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

$3 + 4 + x = 0, \quad x = -7$

$5 - 7 + y = 0, \quad y = 2$

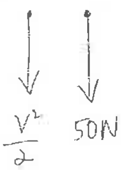
$-6 + 2 + z = 0 \quad z = 4$

The third force is $(-7\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k})$ Newtons

12c

i) $m = 5$ kg $\quad t = 0,\ x = 0,\ v = 12$

$\Sigma F = m\ddot{x}$



$-\frac{v^2}{2} - 50 = 5\ddot{x}$

$(\div 5)$

$\ddot{x} = -\frac{v^2}{10} - 10$

$= \dfrac{-v^2 - 100}{10}$

ii) $\ddot{x} = \dfrac{-v^2 - 100}{10}$

$$\frac{dv}{dt} = \frac{-v^2-100}{10}$$

$$\int_{12}^{v} \frac{dv}{v^2+100} = \int_0^t -\frac{dt}{10}$$

$$\left[\frac{1}{10}\tan^{-1}\frac{v}{10}\right]_{12}^{v} = \left[-\frac{t}{10}\right]_0^t$$

$$\frac{1}{10}\left(\tan^{-1}\frac{v}{10} - \tan^{-1}\frac{12}{10}\right) = -\frac{t}{10}$$

At max height, $v = 0$: $\tan^{-1}\dfrac{12}{10} = t$

$$t = \tan^{-1}\left(\frac{6}{5}\right)$$

iii) $\ddot{x} = \dfrac{-v^2 - 100}{10}$

$$v\frac{dv}{dx} = \frac{-v^2-100}{10}$$

$$\frac{1}{2}\int_{12}^{v} \frac{2v\,dv}{v^2+100} = \int_0^x -\frac{dx}{10}$$

$$\left[\frac{1}{2}\ln(v^2+100)\right]_{12}^{v} = \left[-\frac{x}{10}\right]_0^x$$

$$\ln(v^2+100) - \ln(12^2+100) = -\frac{2x}{10} + 0$$

$$\ln\left(\frac{v^2+100}{244}\right) = -\frac{x}{5} \quad (*)$$

$$\frac{v^2+100}{244} = e^{-x/5}$$

$$v^2 + 100 = 244e^{-x/5}$$

$$\therefore \quad v^2 = 244e^{-x/5} - 100$$

iv) Max height : $v = 0$ into ($*$)

$$x = -5 \ln\left(\frac{100}{244}\right)$$

$$= \left(5 \ln \frac{61}{25}\right) \text{ m}$$

13 a

- Prove that if $(n+2)^2 - n^2$ is a multiple of 8, then n is odd

If $(n+2)^2 - n^2$ is a multiple of 8,

$n^2 + 4n + 4 - n^2 = 8x$ for some integer x

$4n + 4 = 8x$

$n + 1 = 2x$

$n = 2x - 1$

$\therefore$ n is odd

- Prove that if n is odd then $(n+2)^2 - n^2$ is a multiple of 8

If n is odd, let $n = 2x - 1$ for some integer x

$$(n+2)^2 - n^2 = (2x-1+2)^2 - (2x-1)^2$$

$$= (2x-1+2+2x-1)(2x-1+2-2x+1)$$

$$= (4x)(2)$$

$$= 8x$$

which is a multiple of 8.

$\therefore$ $(n+2)^2 - n^2$ is a multiple of 8 iff n is odd

13 b

i) $LHS = \sqrt{\frac{8-x}{x}} \times \frac{\sqrt{8-x}}{\sqrt{8-x}}$

$$= \frac{8-x}{\sqrt{x(8-x)}}$$

$$= \frac{8-x}{\sqrt{8x - x^2}}$$

$$= \frac{4 - x + 4}{\sqrt{8x - x^2}}$$

$$= \frac{4-x}{\sqrt{8x-x^2}} + \frac{4}{\sqrt{8x-x^2}}$$

$= RHS$

ii) $$\int_0^2 \sqrt{\frac{8-x}{x}}\,dx = \int_0^2 \left(\frac{(4-x)}{\sqrt{8x-x^2}} + \frac{4}{\sqrt{16-(x-4)^2}} \right) dx$$

$$= \left[\sqrt{8x-x^2} + 4\sin^{-1}\left(\frac{x-4}{4}\right) \right]_0^2$$

$$= \sqrt{16-4} + 4\left[\sin^{-1}\left(\frac{-2}{4}\right) - \left(0 + \sin^{-1}\left(\frac{-4}{4}\right)\right)\right]$$

$$= \sqrt{12} + 4\left[\sin^{-1}\left(-\tfrac{1}{2}\right) - \sin^{-1}(-1)\right]$$

$$= 2\sqrt{3} + 4\left[\left(-\frac{\pi}{6}\right) - \left(-\frac{\pi}{2}\right)\right]$$

$$= 2\sqrt{3} + \frac{4\pi}{3}$$

13 c

i) $z^7 - 1 = 0$

$(z-1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$

Since α is complex, $\alpha - 1 \neq 0$

$\therefore \ \alpha^6 + \alpha^5 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$

ii) Sum of roots: $\alpha + \alpha^6 + \alpha^2 + \alpha^5 + \alpha^3 + \alpha^4 = -a$

$-1 = -a$

$\boxed{a = 1}$

Sum of roots two at a time:

$(\alpha+\alpha^6)(\alpha^2+\alpha^5) + (\alpha+\alpha^6)(\alpha^3+\alpha^4) + (\alpha^2+\alpha^5)(\alpha^3+\alpha^4) = b$

$\alpha^3 + \alpha^6 + \alpha^8 + \alpha^{11} + \alpha^4 + \alpha^5 + \alpha^9 + \alpha^{10} + \alpha^5 + \alpha^6 + \alpha^8 + \alpha^9 = b$

$\alpha^3 + \alpha^6 + \alpha + \alpha^4 + \alpha^4 + \alpha^5 + \alpha^2 + \alpha^3 + \alpha^5 + \alpha^6 + \alpha + \alpha^2 = b$

$2(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6) = b$

$2(-1) = b$

$\boxed{b = -1}$

Product of roots:

$(\alpha+\alpha^6)(\alpha^2+\alpha^5)(\alpha^3+\alpha^4) = -c$

$(\alpha^3 + \alpha^6 + \alpha^8 + \alpha^{11})(\alpha^3 + \alpha^4) = -c$

$\alpha^6 + \alpha^7 + \alpha^9 + \alpha^{10} + \alpha^{11} + \alpha^{12} + \alpha^{14} + \alpha^{15} = -c$

$\alpha^6 + 1 + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + 1 + \alpha = -c$

$2 - 1 = -c$

$\boxed{c = -1}$

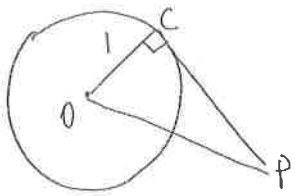
13 d $p = \sqrt{2}$, $q = -\sqrt{2}$

13 e

Assume that pq is irrational, but neither p nor q is irrational, so that both p and q are rational.
Then pq is the product of two rational numbers, so is rational.
This contradicts the assumption that pq is irrational.

14a $\frac{(\log x)^2}{2} + c$

14b



$|\vec{OP}| = \sqrt{1+6+9} = 4$

$|\vec{PC}| = \sqrt{4^2-1^2} = \sqrt{15}$

14c

i) Let $u = x^n$ $\quad dv = \cos x\, dx$

$du = nx^{n-1}\, dx \quad v = \sin x$

$I_n = uv - \int v\, du$

$= x^n \sin x - n\int x^{n-1} \sin x\, dx$ $\quad$ Let $u = x^{n-1}$ $\quad dv = \sin x\, dx$

$du = (n-1)x^{n-2}\, dx \quad V = -\cos x$

$= x^n \sin x - n\left[-x^{n-1}\cos x + (n-1)\int x^{n-2} \cos x\, dx\right]$

$= x^n \sin x + nx^{n-1}\cos x - n(n-1)\, I_{n-2}$

ii) $I_4 = \int x^4 \cos x\, dx$

$I_0 = \int \cos x\, dx$

$= \sin x$

$I_2 = x^2 \sin x + 2x\cos x - 2\sin x$

$I_4 = x^4 \sin x + 4x^3 \cos x - 12\,(x^2\sin x + 2x\cos x - 2\sin x)$

$= x^4 \sin x + 4x^3\cos x - 12x^2 \sin x - 24x\cos x + 24\sin x$

$\int_0^{\pi/4} x^4 \cos x\, dx = \left[x^4 \sin x + 4x^3\cos x - 12x^2\sin x + 24x\cos x + 24\sin x\right]_0^{\pi/2}$

$= \left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24$

$= \frac{\pi^4}{16} - 3\pi^2 + 24$

14d

$|\vec{OA}| = \sqrt{1+3^2+2^2} = \sqrt{14}$

$|\vec{OB}| = \sqrt{4^2+2^2+6^2} = 2\sqrt{14}$

$\vec{OP} = \vec{OA} + \lambda\vec{AB}$

$= \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ -8 \end{pmatrix}$

$= \begin{pmatrix} 1+3\lambda \\ 3-\lambda \\ 2-8\lambda \end{pmatrix}$

In $\triangle OAP$, $\cos\theta = \dfrac{\vec{OA}\cdot\vec{OP}}{|\vec{OA}||\vec{OP}|}$

In $\triangle OBP$, $\cos\theta = \dfrac{\vec{OB}\cdot\vec{OP}}{|\vec{OB}||\vec{OP}|}$

$\therefore \dfrac{\vec{OA}\cdot\vec{OP}}{|\vec{OA}||\vec{OP}|} = \dfrac{\vec{OB}\cdot\vec{OP}}{|\vec{OB}||\vec{OP}|}$

$\vec{OA}\cdot\vec{OP} = \dfrac{\vec{OB}\cdot\vec{OP}}{2}$

$2\left[1+3\lambda+3(3-\lambda+2(2-8\lambda)\right] = \left[4(1+3\lambda)+2(3-\lambda)+6(2-8\lambda)\right]$

$\lambda = \dfrac{1}{3}$

$\lambda = \dfrac{m}{m+n}$

$\dfrac{1}{3} = \dfrac{m}{m+n}$

$m+n = 3m$

$n = 2m$

$\boxed{\dfrac{m}{n} = \dfrac{1}{2}}$

ii) $\overrightarrow{OP} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 3 \\ -1 \\ -8 \end{pmatrix}$ from (i)

$= \frac{1}{3}\begin{pmatrix} 6 \\ 8 \\ -2 \end{pmatrix}$

14e

$$LHS = \frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}$$

since $\sin\theta = \cos(\frac{\pi}{2}-\theta)$, $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$

$$= \frac{1+\cos(\frac{\pi}{2}-\theta) + i\sin(\frac{\pi}{2}-\theta)}{1+\cos\left[-(\frac{\pi}{2}-\theta)\right] + i\sin\left[-(\frac{\pi}{2}-\theta)\right]}$$

$$= \frac{1+e^{i(\frac{\pi}{2}-\theta)}}{1+e^{-i(\frac{\pi}{2}-\theta)}} \times \frac{e^{i(\frac{\pi}{2}-\theta)}}{e^{i(\frac{\pi}{2}-\theta)}}$$

$$= \frac{e^{i(\frac{\pi}{2}-\theta)}\left(1+e^{i(\frac{\pi}{2}-\theta)}\right)}{e^{i(\frac{\pi}{2}-\theta)}+1}$$

$$= e^{i(\frac{\pi}{2}-\theta)}$$

$$= \cos(\tfrac{\pi}{2}-\theta) + i\sin(\tfrac{\pi}{2}-\theta)$$

$$= \sin\theta + i\cos\theta$$

$$= RHS$$

14e

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^5 = -i$$

$(\sin\theta + i\cos\theta)^5 = -i$ from (i)

$$\left[\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right]^5 = -i$$

$$cis\left(\frac{5\pi}{2}-5\theta\right) = cis\left(-\frac{\pi}{2}\right)$$ using De Moivre's Theorem

$$\frac{\pi}{2}-5\theta = -\frac{\pi}{2}$$

$$\pi = 5\theta$$

$$\boxed{\theta = \frac{\pi}{5}}$$

15a

$m = 1700$ kg $\qquad \ddot{x} = 1.2$ m/s^2 $\qquad t=0,\ v=0,\ x=0$

i)

$$\Sigma F = m\ddot{x}$$

$$T - mg = m\ddot{x}$$

$$T = m(\ddot{x}+g)$$

$$= 1700(1.2+9.8)$$

$$= 18700 \text{ Newtons}$$

ii) $t=1.5,\quad v=?\quad x=?$

$$\ddot{x} = 1.2$$

$$\frac{dv}{dt} = 1.2$$

$$v = \int_0^{1.5} 1.2\,dt$$

$$= [1.2t]_0^{1.5}$$

$$= 1.2\times 1.5$$

$$v\frac{dv}{dx} = 1.2$$

$$\int_0^{1.8} v\,dv = \int_0^x 1.2\,dx$$

$$\left[\frac{v^2}{2}\right]_0^{1.8} = [1.2x]_0^x$$

$$1.8^2 = 1.2x$$

$$x = \frac{1.8^2}{2}\div 1.2$$

$$= 1.35m$$

i) $\sqrt{4^2+2^2+4^2} = 6 \, m/s$

ii) $t = 2, \quad z = 0 + 2 \times 4$
$$= 8 \, m$$

iii) $\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix} = -4 \times 4 + 2 \times (-6) + 4 \times 7$
$$= 0$$

iv) $\begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + t \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -39 \\ 44 \\ 0 \end{pmatrix} + s \begin{pmatrix} 4 \\ -6 \\ 0 \end{pmatrix}$

$$5 - 4t = -39 + 4s$$
$$6 + 2t = 44 - 6s$$
$$4t = 7s$$

$$t = 7, \quad s = 4$$

Jordan airplane takes off 3 seconds after Raymond's

15C

i) $C + iS = 1 + \frac{1}{2}\text{cis}\,\theta + \frac{1}{4}\text{cis}\,\theta + \dots + \frac{1}{2^k}\text{cis}(k\theta) + \dots$

$= 1 + \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \frac{1}{8}e^{3i\theta} + \dots$ $\left[\text{G.P } a = 1, r = \frac{1}{2}e^{i\theta}\right]$

$= \dfrac{1}{1 - \frac{1}{2}e^{i\theta}} \times \dfrac{2}{2}$

$= \dfrac{2}{2 - e^{i\theta}}$

ii) $C + iS = \dfrac{2}{2 - \cos\theta - i\sin\theta} \times \dfrac{2 - \cos\theta + i\sin\theta}{2 - \cos\theta + i\sin\theta}$

$= \dfrac{4 - 2\cos\theta + 2i\sin\theta}{(2 - \cos\theta)^2 + \sin^2\theta}$

$= \dfrac{4 - 2\cos\theta + 2i\sin\theta}{4 - 4\cos\theta + \cos^2\theta + \sin^2\theta}$

$= \dfrac{4 - 2\cos\theta + 2i\sin\theta}{5 - 4\cos\theta}$

$C = \text{Re}\,(C + iS)$

$= \dfrac{4 - 2\cos\theta}{5 - 4\cos\theta}$

Q16(a) (i) LHS $= \int_0^a f(x)\,dx$

let $x = a - u$

$dx = -du$

$x = 0, u = a$

$x = a, u = 0$

$= \int_a^0 f(a-u)(-du)$

$= \int_0^a f(a-u)\,du$

$= \int_0^a f(a-x)\,dx$ (u is a dummy variable)

$= \underline{\underline{RHS}}$

(ii) $I = \int_0^{\pi/4} \ln\left[\frac{2}{1+\tan x}\right] dx$

$= \int_0^{\pi/4} \ln 2\,dx - \int_0^{\pi/4} \ln(1+\tan x)\,dx$

$= \int_0^{\pi/4} \ln 2\,dx - \int_0^{\pi/4} \ln\left[1+\tan\left(\frac{\pi}{4}-x\right)\right] dx$

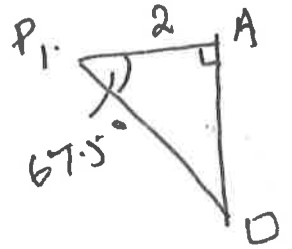
$= \int_0^{\pi/4} \ln 2\,dx - \int_0^{\pi/4} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right) dx$

$I = \int_0^{\pi/4} \ln 2\,dx - \int \ln\left(\frac{2}{1+\tan x}\right) dx$

$I = \int_0^{\pi/4} \ln 2\,dx - I$

$\therefore I = \frac{1}{2}\int_0^{\pi/4} \ln 2\,dx = \underline{\underline{\frac{\pi}{8}\ln 2}}$

16 (b) Let O be the centre of the ~~re~~ regular octahedron $P_1P_2P_3P_4P_5P_6P_7P_8$.



$$|\vec{OA}| = 2\tan 67.5° = 2t$$

$$\tan 135° = \frac{2t}{1-t^2}$$

$$-1 = \frac{2t}{1-t^2}$$

$$t^2 - 2t - 1 = 0$$

$$t = 1 \pm \sqrt{2}$$

as $t > 0$

$$t = 1 + \sqrt{2}$$

$$\therefore |\vec{OA}| = 2(1+\sqrt{2}) \text{ units}$$

$\vec{OP_i}$ $(i = 1, 2, \ldots 8)$ are the 8th roots of

$$z^8 = |\vec{OP_1}|$$

$\Rightarrow$ sum of the roots $\sum_{i=1}^{8} \vec{OP_i} = 0$

$$\Rightarrow \sum_{i=1}^{8} (\vec{OA} + \vec{AP_i}) = 0$$

$$\Rightarrow \sum_{i=1}^{8} \vec{AP_i} = -8\vec{OA}$$

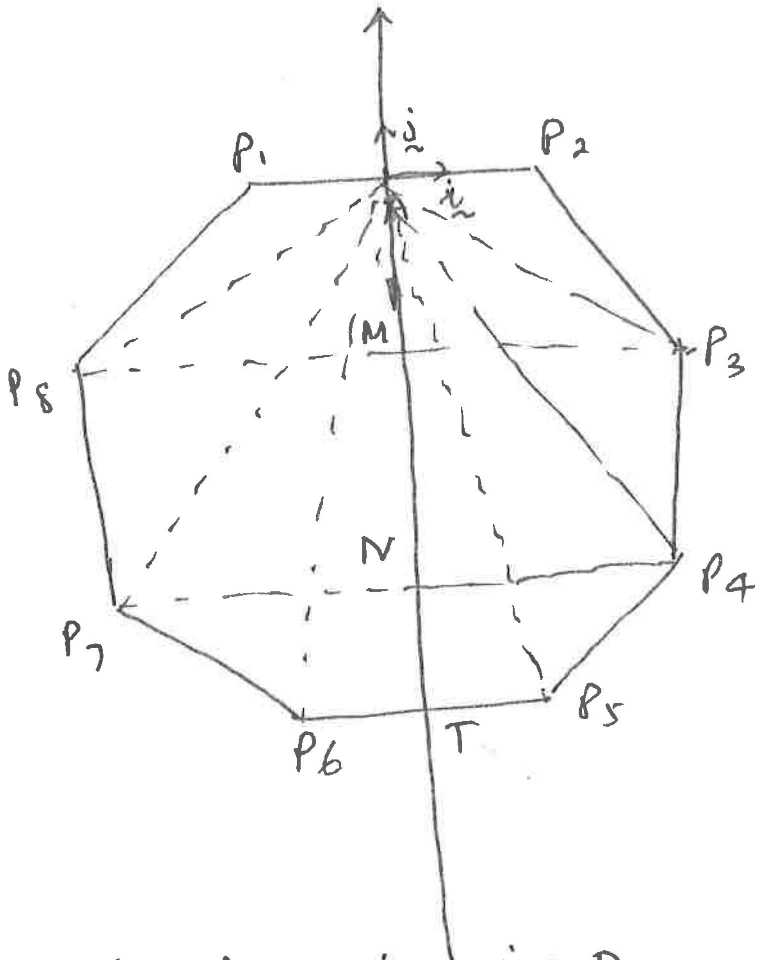
$$\Rightarrow \left|\sum_{i=1}^{8} \vec{AP_i}\right| = 8|\vec{OA}|$$

$$= 8[2(1+\sqrt{2})]$$

$$= 16(1+\sqrt{2}) \text{ units}$$

ie $|\vec{AP_1} + \vec{AP_2} + \vec{AP_3} + \vec{AP_4} + \vec{AP_5} + \vec{AP_6} + \vec{AP_7} + \vec{AP_8}| = 16(1+\sqrt{2})$ units

16(b) Alternative solution 1



$$\overrightarrow{AP_1} + \overrightarrow{AP_2} = -2\underset{\sim}{i} + 2\underset{\sim}{i} = 0$$

$$\overrightarrow{AP_3} + \overrightarrow{AP_8} = 2\overrightarrow{AM} = -4\sqrt{2}\,\underset{\sim}{j}$$

$$\overrightarrow{AP_4} + \overrightarrow{AP_7} = 2\overrightarrow{AN} = -2(2\sqrt{2}+4)\underset{\sim}{j}$$

$$\overrightarrow{AP_5} + \overrightarrow{AP_6} = 2\overrightarrow{AT} = -2(2\sqrt{2}+4+2\sqrt{2})\underset{\sim}{j}$$

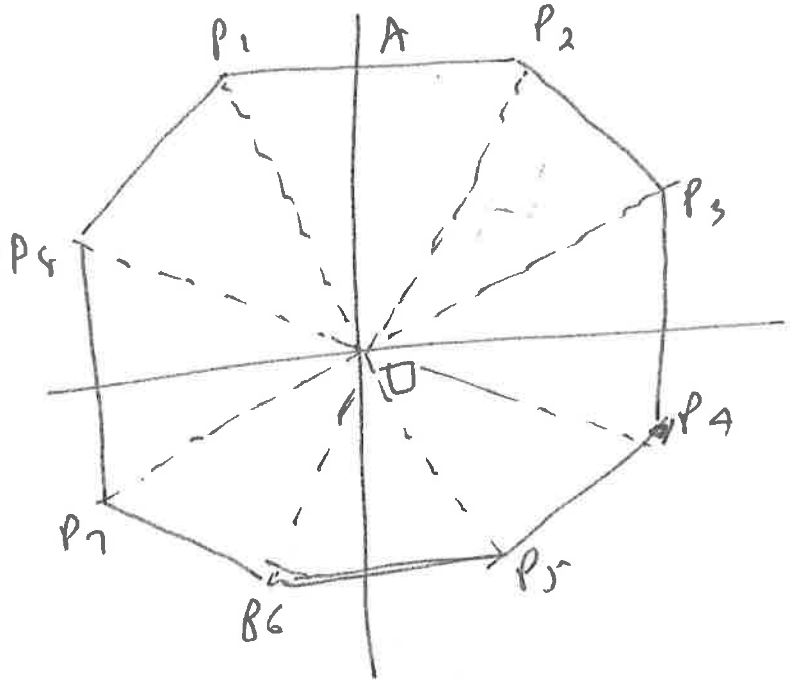
$$\therefore \left|\overrightarrow{AP_1} + \overrightarrow{AP_2} + \overrightarrow{AP_3} + \overrightarrow{AP_4} + \overrightarrow{AP_5} + \overrightarrow{AP_6} + \overrightarrow{AP_7} + \overrightarrow{AP_8}\right| =$$

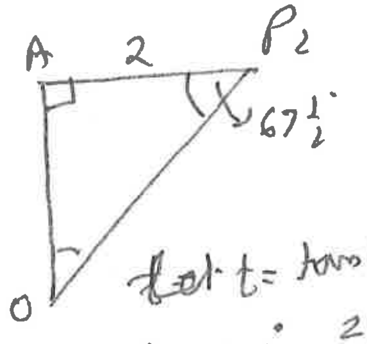
$$\left|\left(\left[-4\sqrt{2} - 2(2\sqrt{2}+4) - 2(2\sqrt{2}+4+2\sqrt{2})\right]\underset{\sim}{j}\right)\right|$$

$$= \left|(-16-16\sqrt{2})\underset{\sim}{j}\right|$$

$$= 16(1+\sqrt{2}) \text{ units}$$

16 (b) Alternative solution 2





Let $t = \tan 67.5°$

$\Rightarrow \tan 135° = \dfrac{2t}{1-t^2}$

$-1 = \dfrac{2t}{1-t^2}$

$t^2 - 2t - 1 = 0$

$t = 1 \pm \sqrt{2}$

$t > 0$

$\therefore t = 1 + \sqrt{2}$

$|\overrightarrow{AO}| = 2t$

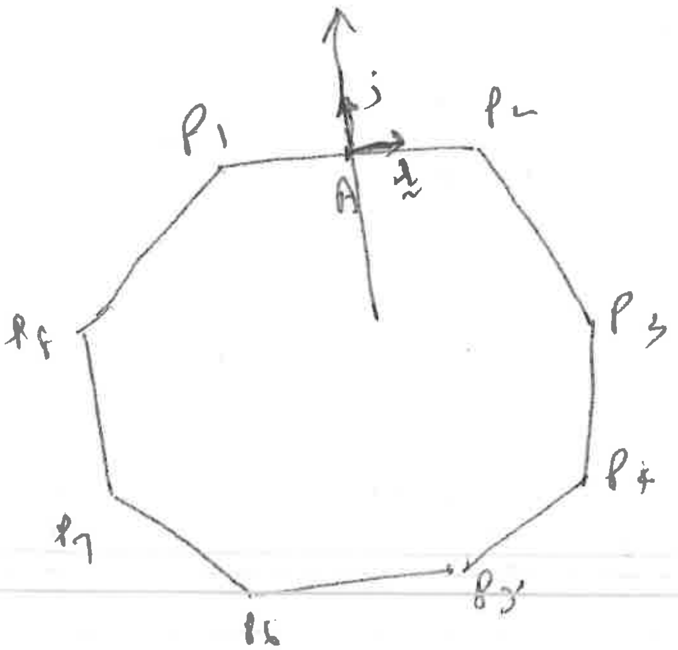
$= 2 + 2\sqrt{2}$

$$\overrightarrow{AP_1} = \overrightarrow{AO} + \overrightarrow{OP_1}$$
$$\overrightarrow{AP_2} = \overrightarrow{AO} + \overrightarrow{OP_2}$$
$$\overrightarrow{AP_3} = \overrightarrow{AO} + \overrightarrow{OP_3}$$
$$\overrightarrow{AP_4} = \overrightarrow{AO} + \overrightarrow{OP_4}$$
$$\overrightarrow{AP_5} = \overrightarrow{AO} + \overrightarrow{OP_5} = \overrightarrow{AO} - \overrightarrow{OP_1}$$
$$\overrightarrow{AP_6} = \overrightarrow{AO} + \overrightarrow{OP_6} = \overrightarrow{AO} - \overrightarrow{OP_2}$$
$$\overrightarrow{AP_7} = \overrightarrow{AO} + \overrightarrow{OP_7} = \overrightarrow{AO} - \overrightarrow{OP_3}$$
$$\overrightarrow{AP_8} = \overrightarrow{AO} + \overrightarrow{OP_8} = \overrightarrow{AO} - \overrightarrow{OP_4}$$

Adding $\left|\overrightarrow{AP_1} + \overrightarrow{AP_2} + \overrightarrow{AP_3} + \overrightarrow{AP_4} + \overrightarrow{AP_5} + \overrightarrow{AP_6} + \overrightarrow{AP_7} + \overrightarrow{AP_8}\right| =$

$$\left|8\overrightarrow{AO}\right|$$
$$= 8\left|\overrightarrow{AO}\right|$$
$$= 8(2 + 2\sqrt{2})$$
$$= (16 + 16\sqrt{2}) \text{ units}$$

16(b) Alternative Solution 3



$$\vec{AP_2} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \cancel{\vec{AP_1} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}}$$

$$\vec{AP_3} = \begin{pmatrix} 2+2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix} \qquad \cancel{\vec{AP_8} = \begin{pmatrix} -2-2\sqrt{2} \\ -2\sqrt{2} \end{pmatrix}}$$

$$\vec{AP_4} = \begin{pmatrix} 2+2\sqrt{2} \\ -2-2\sqrt{2} \end{pmatrix}$$

$$\vec{AP_5} = \begin{pmatrix} 2 \\ -4-4\sqrt{2} \end{pmatrix}$$

$$\therefore \vec{AP_2} + \vec{AP_3} + \vec{AP_4} + \vec{AP_5} = \begin{pmatrix} 8+6\sqrt{2} \\ -8-8\sqrt{2} \end{pmatrix}$$

By symmetry

$$\vec{AP_1} + \vec{AP_8} + \vec{AP_7} + \vec{AP_6} = \begin{pmatrix} -8-6\sqrt{2} \\ -8-8\sqrt{2} \end{pmatrix}$$

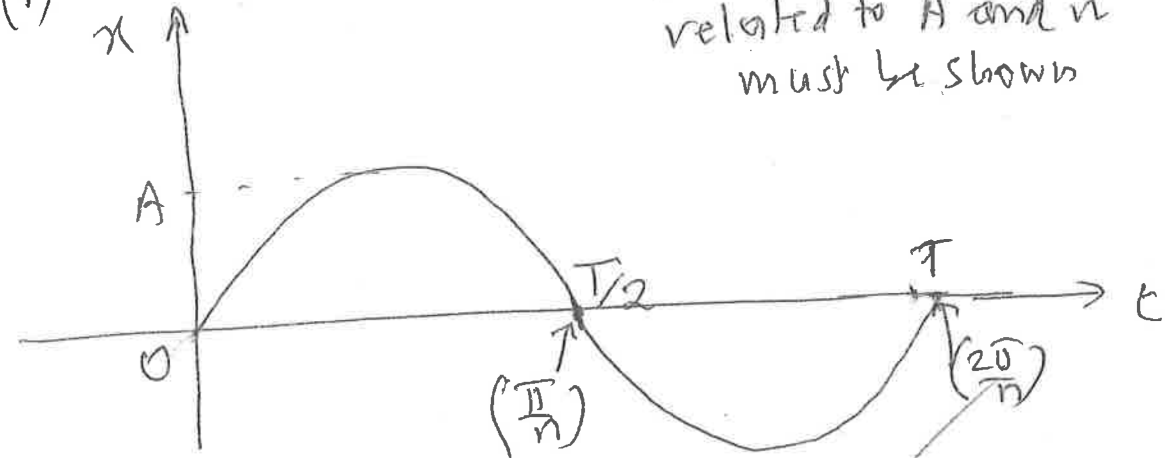
$$\therefore \left| \vec{AP_1} + \vec{AP_2} + \vec{AP_3} + \vec{AP_4} + \vec{AP_5} + \vec{AP_6} + \vec{AP_7} + \vec{AP_8} \right| = \left| \begin{pmatrix} 0 \\ -16-16\sqrt{2} \end{pmatrix} \right|$$

$$= (16 + 16\sqrt{2}) \text{ units}$$

16(c)

$\underline{x = A\sin(nt)}$

(i)



• Points on the graph related to A and n must be shown

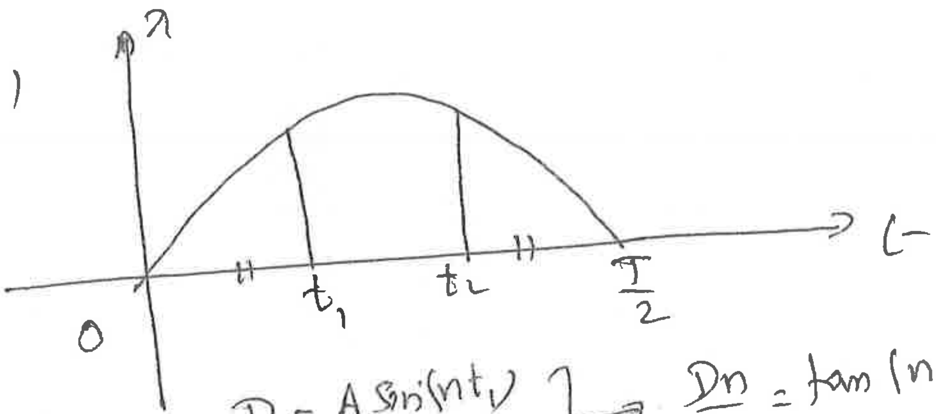
- Axes labelled
- Amplitude shown
- Intercepts $\frac{\pi}{n}$ and $\frac{2\pi}{n}$ shown

(ii)

$$x = A\sin(nt)$$
$$\dot{x} = An\cos(nt)$$
$$= \underline{\frac{2\pi A}{T}\cos\left(\frac{2\pi t}{T}\right)}$$

$$\frac{2\pi}{n} = T$$
$$\therefore n = \frac{2\pi}{T}$$

(iii)



$$\left.\begin{aligned} D &= A\sin(nt_1) \\ V &= nA\cos(nt_1) \end{aligned}\right\} \Longrightarrow \frac{Dn}{V} = \tan(nt_1)$$
$$\Longrightarrow t_1 = \frac{1}{n}\tan^{-1}\left(\frac{Dn}{V}\right)$$
$$= \frac{T}{2\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)$$

The time between the first two occasions when the particle passes through P $= t_2 - t_1$

$$= \left(\frac{T}{2} - t_1\right) - t_1$$
$$= \frac{T}{2} - 2t_1 = \frac{T}{2} - \frac{T}{\pi}\tan^{-1}\left(\frac{2\pi D}{VT}\right)$$
$$= \frac{T}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{2\pi D}{VT}\right)\right]$$

$$= \underline{\frac{T}{\pi}\tan^{-1}\left(\frac{VT}{2\pi D}\right)}$$